On Introducing into the Foundations of Physics the Notions of "The Probing of, Approximation to and Idealization of Structure" *

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Abstract

The first event in the history of a universe, or more accurately, any minimal event in an observer's history should not be burdened by the imposition of unwarranted structure. From this position, we attempt to see an evolving universe as one that is also dramatically evolving in its structure. Structure is not seen as an absolute, but rather consisting of properties that can be gleaned from a local environment. Thus, the first event is seen as a morphism in a category with only one morphism. It is also seen as a minimal element in some Domain. Your local lab is a place to interpret the events of history in terms of local events. Local events are also seen as both elements of a Domain as well as morphisms in a category. Early events are mapped to local events via both a functor and a Domain map. This pairing of Domain maps and functors is meant to draw the notion of a continuous functor into the foundations of physics. Finally, we broach the subject of how idealizations in mathematics (where *the category* of Sets is seen as an idealization) can be seen as the result of iteration and approximation. This deeply impacts our basic reasoning about a universe, since we can dispense with the notion of a single, immutable universe of Sets for a notion of an asynchronous system which, via evolution, approximates the idealization of the category of Sets. Finally, we consider applications to the problem of indefinite causal structure.

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1 Introduction

Perhaps the most fascinating and intellectually challenging disciplines in physics is that of quantum gravity. No other program embodies the speculation and risk that is truly at the root of science. Science structures our contact with nature that we might take risks in discovering its secrets. We rely on the methods and ideas of the past as a means of guiding us safely. It is, therefore, somewhat frightening to dispense with a framework you have been trained to use to interpret and organize the world around you. It is also exhilarating. The rewards are far greater than one can expect to get from using the ideas that have already been tested upon the thresher of mother nature.

Our notions of the interplay between mathematics and theoretical physics must be understood as primarily concerned with the foundations of both mathematics and in physics. A proper attempt at expressing this interplay comes from recognizing various viewpoints in the foundation of mathematics as being similar to those in physics and vice versa. Unfortunately, there occasionally occurs moments in the evolution of our knowledge, where neither realm has completely solidified itself. A trained physicist can only draw upon his intuitions and hope that he will recognize mathematics which expresses his viewpoint. Whereas a mathematician may dream of some form of mathematics, perhaps come to the conclusion that this might have something to do with reality, and be lucky enough to find that group of researchers who are dealing with just such aspects of reality. In any case, the interplay should enrich both disciplines.

Though it would be nice if one could read concurrently about the foundational aspects of mathematics and physics which I will discuss here, we will have to be satisfied by the sequential nature of the printed word. A bit of sightseeing in either direction will help expose to the reader the conclusions of the author.

2 Related Work

The work herein is based on both foundational beliefs in physics and mathematics, as well as some existing work in mathematical physics. Interesting work on the notions of locality, realism, and the epistemic nature of quantum mechanics can be found in [7], [6], [15]. Questions regarding realism in quantum mechanics can be addressed by adopting the view from quantum gravity that processes are the only mathematical objects worthy of being said to "exist" in any sense. This combines the philosophy one sees in works like [14, 8]. This is further abstracted into the categorical models of quantum mechanics we see in [3], [1].

Herein, we dispense with a view of spacetimes as manifolds preferring instead a background of causal structures and their structure preserving maps. This has been discussed in several places, and of note are [12], [9]. Some founding works on incorporating the philosophy that physics and mathematics are done decidedly *within* a universe or category are [4], and [10]. An excellent theory which draws many of these notions together and identifies composition in a category with the causal relation between events can be found in [2].

Briefly, if one wants to see some of these philosophies expressed in simple terms by a mathematician one should start with [11]. Also, one can see some of this philosophy being used by a biologist in [13].

3 Discretization

We begin with a brief talk about discretization. Beginning in the eighties, a program was developed to discretized geometry in the hopes of creating a quantum gravity. Since gravity, according to general relativity, is an effect of the geometry of space-time, quantizing this force meant quantizing space-time itself. What you will find here is one layer beneath this program. This paper is the culmination of an attempt to understand the discretization of the continuum. It can be seen as one layer below the geometric version if we see a manifold as a topological space with added, differentiable structure. The first thought that one might have, in terms of discretizing the continuum are Domains. The analogy is simple. The universe is a causally ordered "set" of events which evolves to a new causally ordered set of events. We can even have a notion of continuity and that is the Scott Topology. Here we have a topology defined on an ordered structure. The problem is that, given that causal structure may be indefinite, we may not be permitted to speak about a set of events for the universe. This problem is handled in subsequent sections.

There is a very deep reason to be concerned with the discretization of the continuum. It is likely that thought experiments in physics rely on our notions of the continuum. Thought experiments should remain an important tool for the intuitive bond a physicist should have with the world around him. However, as we have seen, the continuum in terms of the real numbers has been identified as a gateway to fixed backgrounds [8]. This is a stumbling block on the side of quantum theory on the path to a quantum gravity. By discretizing the continuum, or at least taking it apart with modern mathematical tools, we may be able to preserve the presence of intuition while at the same time dispensing with fixed backgrounds.

4 The Foundational Path

Next, I would like to provide a pre-amble on intuition and the presentation and probing of structure. Intuition seems to be somehow, vaguely related to our notions of the continuum. Continuity is understood as a property held by the maps in the category of topological spaces. We can define various topologies and even sets that look discrete. For instance, given the set of all algebraic structures of some type (groups for instance) we can define an algebraic-dcpo where the elements are structures ordered by sub-structure. We can also define a Scott topology on this dcpo. The bottom or minimal elements of the dcpo would be those with no structure but the structure that is common to all structures of that type, namely the trivial group. The maximal elements would be the free groups where no two elements are the same. A dcpo-map might take us from the dcpo $\{I\}$ containing only the trivial group, to a dcpo

$$I < G$$

where G is any group and thus I, being a subgroup of G, is less than G.

At this point, it will be helpful to consider a fundamental question in the physics of the early universe. Consider for a moment the very first event. We call this event the big bang, or more precisely, the beginning of the big bang since the bang typically also refers to the inflationary period. One might protest and suggest that the universe was always infinite, but the notion of a first event does not betray this sentiment. The reason for this is that events need not be causally related. In my case, sitting here in my apartment, there is both a history for me that goes back to some first event, or set of minimal events, and there are also causally disjoint parts of the universe which actually share no history with me at all. Those parts also have their respective first events. Since we are acausal, there is no sense in attempting to say which of our minimal events is actually the minimum, or first, event overall for the entire universe. The bulk of inflationary theories accept that a first event is untenable, and that, in some sense, the universe was infinite at all phases. Thus, the work here does not betray this.

The important question, however, is not which event was first in the history of the universe. Far more important is the question of what kinds of structure we should justifiably ascribe to our minimal events. Is it reasonable to assume that the first event had any of the structure that events in our current universe, and more specifically any local laboratory, can clearly be observed to possess? For instance, suppose we give ourselves a framework that asserts that all events are quantum in the sense that all events are morphisms in the category of Hilbert spaces. Should we then assume that the very first event should definitely be imparted with all the structure of a morphism of a Hilbert space? Given no other systems or processes with which to interact, how can one tell what the structural properties are?

It is possible that physicists typically impart this kind of structure to the earliest events simply because, mathematically speaking, it is more difficult to work in a frame that is changing. There are, in fact, far more convincing reasons not to impart early events with the structure that one can determine of a local lab setting. Since we are observers who are decidedly *within* a universe, it is far more pleasing philosophically to accept the fact that our measurements are made, not just inside the universe, but inside a carefully constructed laboratory. To put a point on it, we interpret our histories via morphisms in our local lab. These morphisms can be used to probe the structure of earlier events, and thus there is no need to assume any early universe structures.

Returning to mathematics, we need to find fundamental methods in mathematics which reflect this need to probe structure with local categories. So far, we have seen that there is a good foundation for the notion of continuously evolving structure. However, this hardly looks anything like the philosophy of physics outlined above. We need more than just dcpos of algebraic structures. We need to turn to categories. However, we will not want to dispense with what we know about dcpos of algebraic structures. It is imperative that we preserve a notion of the continuum and this seems to mean abstracting the notion of Scott topology up into the realm of categories. We need to be able to smoothly reason over all structures. What is the simplest path from dcpo's of algebraic structures to continuous functors? It would be nice if we could simply see the axioms of the algebraic structures as also axioms of categories.

It is helpful to now discuss approximation and idealization in mathematics and computation. Approximation in computer science is understood as iteration. Iteration, in turn is given meaning via the semantics of programming languages. Algebraic dcpos have been found to be useful in computer science as they provide a semantics for various computational effects such as iteration. It is well known that in the setting of the dcpos of algebraic structures mentioned above, the finitely generated structures are precisely the compact elements of the dcpo.

Our fondness for familiar tools of mathematics sometimes hobbles us. We become dependent upon them and our reliance on them can, if we are not careful, become part of the foundations of physics, when no real justification was apparent. Of course, our justification for relying on differing aspects of mathematics comes, and rightly, from our repeated observation of their ability to allow us to predict the outcomes of measurement. In modern physics, there are several who have attempted to remove our reliance on certain forms of mathematics, specifically to avoid its unjustified presence in the foundations of physics. Though I sympathize with this program, it is our prerogative as physicists to consider the possibility of providing us a way to see mathematics in the foundations of physics, without the problems of being hobbled by fixed frameworks. The philosophy that mathematics is not an aspect of nature is the positivist philosophy and rightly has a place in modern physics. However, it may unnecessarily subjugate the physicist (with his thought experiments) to a learning machine with nothing but data structures in which to hold information used to predict the results of experiment.

4.1 A Motivation for Approximation and Idealization in Physics

The key to understanding how to place mathematics, and especially intuition, back into the foundations of physics, without fear of unjustified backgrounds and frameworks, is through the careful understanding of causal structure, the nature of observation, and the pragmatic use of approximation and idealization. The main intuition for motivating the pragmatics of approximation and idealization, is by imagining systems which are distantly located, and attempting to communicate in the presence of black holes. These communicating processes are trying to establish a notion of the Universe. This is somewhat like the consensus problem for group communication systems [5]. We see each process as a transformation of a system of type A, and the communications between systems of type A as the causal relations between those systems. In Hardy's language, the arrows are like pointing apertures at apertures [6] or connecting the wires of some box or device. We think that the ability to send a message to a system is the same as being causally connected to that system. If a system drops into a black hole, we are no longer causally connected to that system. However, imagine that we have sent a message to a distant receiver, expecting to receive an acknowledgment (a message saying the message was received). From our perspective, we are waiting for the acknowledgment but it doesn't come. The local clock of the distant system can be slowed arbitrarily, while maintaining a causal connection. This means that, from our perspective, we cannot tell the difference between the case that the receiver fell into the black hole and simply having its clock slowed. We are in a situation of indefinite causal structure.

The message to take away is that, in the face of indefinite causal structure, certain kinds of knowledge or reasoning are untenable. Specifically, the set of all processes to which an observer is causally connected is untenable. Recall that the group of receivers were trying to establish a notion of the Universe. Unfortunately, they cannot have a definite notion of causal structure. Unfortunately, this also means that Set structure is also untenable. An observer in this kind of situation is not really free to use the category of sets to talk about his universe. Here we have a stark proof that certain uses of familiar mathematics can prove to be a crutch that must be dispensed with if we want to consider certain important cases in physics. In this case, we have indefinite causal structure, and all of the category of Sets needs to be dispensed with.

There is a way out of this predicament. We know that in a universe, or local universe, where causality does not undergo such shocking changes of state, the use of the category of sets to talk about our local universe is perfectly valid. What is the connective tissue that brings us from the world where Set is untenable, to where set is tenable? This is precisely idealization and approximation, and it is also precisely the evolution of the universe itself.

Recall that we were attempting to see the evolving Universe as also an evolving structure. We are now at the point where we would like it if the universe could evolve to the point where it could support set structure. This is unreasonable when there was only one event, but what if there were many trillions of events?

5 Approximation and Idealization of Structures Themselves

Accepting a notion of iteration and approximation in or around the foundations of physics should come as a bit of a thorn to any physicist. After all, these notions concern the business of constructing something out of something else much like a wood worker can construct a toy soldier out of wood. The toy solder is not a soldier, it is a mock up of one. This is precisely the same way one may view a Java package that allows one to "work with vectors". A piece of software that allows one to instantiate (create) vectors and then add them or find their norm can be seen as a kind of toy soldier. It is not *the category* of vector spaces, yet we can *have* vectors and transform them just as we might on paper like this:

$$O : v \to v'.$$

In order to proceed, one must understand that, in exactly the same way that we can compute π using an algorithm that approximates it and approaches it, we can approximate and approach the category of vector spaces. Imagine that you are the one writing the vector-math package. You start with just a blank text file and start typing words. At first, your program does nothing but slowly you are able to create an object called v and you can find its components, but you cannot add v to anything else. Next you add the ability to add v and w, and you test your program and find it works. Next you implement the ability to find the norm of v but it doesn't quite work. You run the program and it doesn't give quite what the norm should be so you adjust some of the words in the text file until *bingo*, you can now properly compute the norm of v.

What you are doing is precisely iterating. The question is, what are you iterating over? In the case of computing π, you are iterating over the rational numbers. Now, it seems you are iterating over entire structures and really you should also be iterating over categories. Once you are finished, any vector v that you instantiate will have all and exactly those transformative properties of the familiar objects of the category of Vector spaces. The question is, what was the program before you got it working?

The answer to this can only be understood in terms of what can only be called finitely generated structures. In the same way that in the dcpo of groups, finitely generated groups are the familiar ones, when iterating over categories, the familiar categories are those that are finitely generated or perhaps better phrased as fintely presentable. We know that familiar categories are presented with some finite set of axioms which describe universal properties of all the morphisms of that category. Thus, the familiar categories are the finitely presentable categories when we look at each category as being an element of some kind of dcpo. When looked at this way, all the intervening steps we took during our programing exercise of creating a vector math package were most likely some category without a finite presentation. The program did something, but because we have no finite theory for what it did, it simply has not shown up in mathematical literature. Categories without finite presentations are not familiar, nor are they friendly nor even useful except as connective tissue for understanding a process of smoothly traversing a category of categories.

5.1 How a Physicist Should View Categories

Perhaps the best way to see a category, for the purpose of having and refining a physical theory, is to simply see it as a very large set of morphisms along with a very large set of data which describes simple algebraic relations among the morphisms. For instance, there are categories in which each morphism has an inverse. There are two ways to have a universal view on this category. First is simply to state that every morphism has an inverse. That is the standard axiomatic way. The other way is to have a huge collection of morphisms and a large table of data which, upon exhaustive inspection, would reveal that each morphism comes with an inverse. Suppose that one of the morphisms did not have an inverse. How dreadful and disgusting! Our beautifully succinct axiom, that all morphisms have an inverse, is no longer true all because of one morphism. The category in which all morphisms have an inverse is an idealization and is useful and appears in mathematics exactly because of this!

This category is still fairly well behaved in that we still have finitely many things to say about it, namely all but one morphism has an inverse and the one without an inverse is x. Thus, it is still compact in a sense. That sense of compactness is again the notion of idealization and it is also an abstraction of the notion of compactness we see in Domain theory except now we are talking about category structures. There is a vastness of categories, they simply don't have finite sets of axioms. This is the connective tissue of the continuum of categories.

5.2 The Categorical Semantics for Physics

As I have mentioned, we take the view here that all knowledge is gleaned in a local lab. The lab is taken as some system of structure category X. For example, a telescope is a local lab and consists of the Cartesian product of several variables to which the scientist has access. Specifically, a telescope might have two axes of rotation and a focus knob. The category that describes this local lab should support a Cartesian product and we see the lab as something like $F \times \omega \times \theta$. Where F is a space for the degrees of freedom of the focus knob, and θ and ω are the angles. Suppose the scientist has his telescope focused on a distant amorphous gas cloud. He then turns his knob and brings into focus objects closer to earth. He suddenly sees spiral galaxies rather than gas clouds. The turning of the knob is understood as some morphism $f : F \times \omega \times \theta \to F \times \omega \times \theta$ and is a structure preserving map of the system.

Thus we have some local morphism f and we want to use it to talk about what we think of as distant to us. In other words, we want to use it to probe the properties of the system we are observing. As pointed out, the scientist saw an amorphous gas cloud become a spiral galaxy. This too can be seen as some morphism of a structured system. Let the gas cloud be A and the spiral galaxy be B and the transformation is $g : A \to B$. The scientist is trying to understand the distant system by his local lab transformations and so he puts the two transformations side by side as in figure 1.

Next, the scientist must now draw the conclusion as to the relation between the arrows. So he simply completes the diagram like in figure 2.

The scientist now has the functor $U : C \to L$, where C is some imprecisely known category and L is the local lab category. The scientist is probing the structure of the unknown category with his local lab. One can now see quite

$$
\begin{tikzcd}
A \arrow[d, "g"] & F \times \omega \times \theta \arrow[d, "f"] \\
B & F \times \omega \times \theta
\end{tikzcd}
$$

Figure 1: The Arrows are put side by side

$$
\begin{tikzcd}
A \arrow[r, "U"] \arrow[d, "g"] & F \times \omega \times \theta \arrow[d, "f"] \\
B \arrow[r, "U"'] & F \times \omega \times \theta
\end{tikzcd}
$$

Figure 2: We complete the square with the Functor U

clearly why interpretations of quantum mechanics have been so difficult. A Stern Gerlach experiment can be cast in exactly the same way where there is some strange system (nuclei coming out of a source) being probed by some classical apparatus (magnetic fields controlled by knobs with classical data output). The category of the apparatus is clearly a category with a Cartesian product, but the imprecisely known system is best understood as an object in a symmetric monoidal category. We would have trouble probing out the tensor product structure with the given apparatus.

5.3 Further Localization of Physics

In this section we are positing the existence of a distant system and that its transformations can, in a way, induce transformations in our local lab. To a degree we described the gas cloud becoming a spiral galaxy as if these were objects to which we had access. Of course, they are not. Furthermore, when we “see” the gas cloud become a spiral galaxy, we are actually observing a transformation of our local lab. The view through the eyepiece has changed.

Thus, to be fair, both arrows f and g are actually in the local lab. So, what is going on here?

We are faced with a rather frightening notion, namely, that what we are "seeing" in our lab is not the object under study, but a mimic of it constructed in the local lab. After all, the only morphisms that we can take as being real are the turning of the knob and the change in the appearance of the eyepiece. These are strictly changes of the local lab. We have to accept that we only have an *intuition* of the distant or mysterious system and we are using diagrams in the local lab to mock up, or probe, diagrams in the distant system.

What I have begun to describe here, is nothing other than topos theory. Topos theory is simply the answering of the question: "If I can only talk about things in a single category D, like SET, how do I ensure that I can talk about everything I want to?"

I will include, here, a brief, rough description of the mathematics which seems to underly this view of physics.

5.3.1 Doing Math in Category D

Suppose I want to establish some property of C, but what I really have is D, some precisely known category. I would normally have diagrams in C by mapping cpos or Domains into C. But instead, I want to do it with D. What I do is define the largest category, J, of Domains in D. I do this by defining a dcpo with objects as elements in D and relations as arrows in D. Then any functor will map the domains in J to diagrams in C. It seems like I am just inserting a category D in the normal diagram functor $J \to C$ resulting in $J \to D \to C$ which seems to miss the point of the exercise. The point of the exercise, is to try to do a lot of category theory when you have to live in some category D. We start by saying that we "have access to" all diagrams in D. Next, we intuit the existence of a category C (I am using this restricted language to reflect the notion that we do don't have access to C). Next, we consider endofuntors of D, but we really see them as diagrams in D indexed by the domains we constructed in J. These endofunctors are meant to mimic functors from D to C. We are pretending to have access to C, by attempting a construction of C in D. There are a few questions:

1. What kind of minimum structure do we need in D to really start doing some work?

2. If we really want to say that we only have access to D, then we cannot present D as a set of morphisms and a set of objects because that would imply we are actually in SET, not D. Is there any way to start working only in D? This goes back to the first question (although thinking about this too much is a bit of a morass).

The answer is that D must have all finite limits.

5.4 A Hands on Application of Approximating Structures in Physics

We are still left wondering where the line is between constructing things that mimic our idealizations (through the process of iteration), and what constitutes ultimate reality. It might be clear that I am tentatively operating under the assumption that the only thing that constitutes ultimate reality is a causal structure and the way causal structures can evolve (the domain map). To further illustrate how deeply we find iteration and approximation in real, hands-on physics, I will now paint a picture of an optics lab in terms of approximation.

An optics lab typically consists of an optical table, a laser and mirrors (among other things). If we are going to discuss foundations, we need to go deep into the construction of the lab itself. Thus, we begin with the extraction of ore from the ground. The ore is dug, and separated from soil. This, itself, is a purification process. The ore is then melted and the iron is then skimmed off. Again, another refinement process. We now have an impure form of iron, or an approximation to an idealization known as iron. Next, the iron is poured into a mold of the surface of the optical table. It is allowed to cool and then it is rolled. Next its surface must be machined flat. This has, in the past, been accomplished with the use of two tools. First is the grinder, and second is a flatness reference which consists of a blue oil and a precision metal bar. The surface of the precise reference-bar is coated with oil and swept across the table. If the oil does not coat the table uniformly, then the table is not as flat as the reference and must further be machined. Back and forth, one machines and then tests with the reference. Eventually, the table is as flat as the reference.

In terms of a dcpo, the reference is the finite element and the table must get beyond this element in the dcpo. Thus, in general, references are the finite or compact elements of the dcpo. It was a finite process or a finite expression that describes the construction or the qualities of the reference. These qualities can now be passed to the surface of the table. In another sense, the properties in the reference have been copied to the table.

What is it that we are approximating? In fact, we are constructing reference frames, and in a way, we are approximating the category of vector spaces. Upon completion, we can say that the normal to the surface of the table is sufficiently similar at all points on the surface of the table. Thus, there is a three dimensional coordinate system, up to a rotation, that is shared at each point on the table. The use of iron was very important since if the table flexed, we could not say that we had the same frame at all points at all times. Iron does not flex easily if it is thick enough.

Next, we screw a translation stage into one of the holes of the optical table and bolt a laser to the stage. Further, we screw down a mirror a few feet away from the laser. The surface of the mirror is also a frame defined by the normal to the mirror and the laser is also a frame defined by the direction of the beam. These frames are most likely higher up than the normal defined by the surface of the table. That is to say, the frame defined by the beam of the laser (low dispersion defines a precise frame), is more precisely defined than the frame of the table which will have some variation from point to point. This is where the translation stages come in.

We have one, roughly approximated frame over the surface of the table, but it is not as precisely defined as the surface of the mirror and the beam of the laser. Thus, once the beam is turned on and pointed at the mirror we can now use the translation stages to make the beam of the laser have precisely the same frame as the surface of the mirror. Perhaps, the mirror only reflects a certain wavelength. This light, if the frames are exactly the same, will travel exactly back along the beam of the laser and even enter the laser and deplete the gain medium at a precise wavelength. The mirror is actually selecting a more precise wavelength from the laser.

In short, the experimentalist is further refining something right in the lab *during* the experimental preparation. He is iterating towards a point where the laser and the mirror share some unknown, but more precisely equivalent frame. He is also further refining the wavelength of light in the laser. More importantly, though, he can now successfully use the category of vector spaces to talk about what is going on in the lab. Through careful preparation and refinement, a kind of cultivation of the local environment, he now has a system that, much like the Java vector math package, really behaves like the category of vector spaces.

6 Conclusions

Foundational notions in mathematics can be reflected in foundational notions of physics and vice versa. The author has avoided taking well known formulae and methods in computing science and mathematics and gone about finding their semantics in any domain that could be considered physical. Instead, the paper is quite the opposite. Some very basic aspects of the day to day practice of physics have been abstracted into some fundamental notions of mathematics with the hope that the vast amount of structures that can be constructed will lead up towards the kinds of theories physicists are used to working with. This work is, at its heart, a foundational exercise in physics with the intent that the chosen set of salient beliefs in physics, being placed as close to the core of doing mathematics as possible, will be reflected at all stages of construction. Furthermore, care has been taken so that any abstraction borrowed from the theories of computation and mathematics are well reflected in only the most pragmatic aspects of the day to day job of a physicist, rather than any existing theory of physics.

It should be made clear, for the untrained mathematician and physicist, that a solid mathematical foundation is lacking within this paper. Rather, it is the drawing together of some well known ideas, and the expression of some very vague and weak notions in mathematics and physics. A solid foundation is expected to appear, and some progress towards that has been demonstrated.

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