THE ENTROPY OF TANGLES

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Abstract. A means of looking at tangles, the familiar messes which we pull from kitchen drawers, is provided. By identifying words in countable subgroups of the set of morphisms of a string with knot classes one can generate a statistics. From here an entropy is straightforward. The work is incomplete and requires the input of a knot theorist to help define the identification between diffeomorphisms and knot classes.

1. Those Frustrating Tangles

How many times have you removed your walkman, mp3 player or yarn ball from a desk drawer or knapsack? Inevitablly, unless you have been very carefull, these strings come out tangled, leaving you with a rather vexing problem. How do I untangle this mess? While the tangle which we are familiar with is not a knot in the formal sense, it is very similar and has many of the same properties.

In the real world, when we come upon a tangle or a knot, we have no idea how it got that way. If we had taken down the exact sequence of twists and loops that were enacted upon our string, we would hardly even consider it a tangle. Grasping it from the drawer we would immediately apply the reverse set of moves that got it tangled in the first place. Thus we may see our tangle in a new light.

When the string was placed in the drawer, its state was pure, statistically speaking, in that we knew it was untangled. So, we can see a tangle as a random number of morphisms of an untangled string, and we can also see it as a probability distribution over equivalence classes within the set of morphisms, or a probability distribution over the set of knots.

A mathematician who is given a knot will attempt to provide the equivalence class of knots in which it sits. He might even try to calculate an invariant from the given knot and then use that to provide the type of knot. From a more modest perspective, one might just want to know how to get our knot back to being a string, or how to go from a string to a knot.

2. The Space of Morphisms

The endpoints of our string are free and thus every tangle is really just a string or interval. For purposes of calculation, we need to embed our string as a curve in three dimensional space. Furthermore, let us connect the ends of the string to the opposite ends of a sphere so that the string sits inside that ball. Were we to glue the ends of the string together, then we might have a circle or a trefoil knot or some other knot.

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In the real world, the string underwent a different morphism each time we opened the drawer to get the scissors or the tape until finally, if we were to glue the ends of the string together, we would have a knot, or not. Thus the set of morphisms of an interval can be partitioned into equivalence classes if we finish each morphism with an identification of the endpoints of the string. The combined set of morphisms, formed from the disjoint union of morphism of the interval plus the identification of the endpoints will be called M. Certain morphisms map the interval to a circle and I will call this equivalence class of morphisms the identity I. We can identify each of the remaining equivalence classes M_i with a unique element of the set of knot equivalence classes K_i. If we apply a morphism from M_i and identify the ends of the string, then the resulting knot is in the equivalence class K_i.

One might jump right past the morphisms and cut right to the set of knots. Recall, however that in reality, we are standing in the kitchen, the tangle of string still dangling from our fingers as we struggle to understand the scientist who is explaining knot theory on a blackbord.

"Wait, wait." We have to ask "How will we eventually understand how to untangle the tangle?"

"This will require knowing and applying a special set of morphisms." says the scientist.

"Oh. Ok. Well please go on."

3. The Statistics of a Tangle

To come to the point where we can start talking about tangles thermodynamically, we have to know where our statistics come from. Just as there was a set of morphisms or moves which transformed our nice piece of string into a tangle, the reverse set of moves will transform our tangle back into a string. Unfortunately, we did not keep track of the morphisms which were being applied to our tangle. In this sense, the tangle very much resembles an ideal gas, simply because the exact state of the tangle is not known. Even though we can see exactly what the state of the string is, because we are looking right at it, we still don't know what moves we are going to have to take to get it back to being untangled. If we *did* know which moves to employ, we wouldn't even really consider it a tangle. Our shoelaces, for example, are not considered tangles since, most of the time, if we identified the ends, and then tugged on the resulting "tangled" loop, it would untangle itself.

The space of morphisms M is an algebraic structure with manifold structure. However, the actual form of algebraic structure is not clear. Furthermore, we can envision an infinite subroup $G \in M$, a subgroup that is dense in M, which is generated by some finite set of moves. This subgroup G is a countable subgroup and it has generators $a_i \in G$. Call the set of generators $S|a_i \in S$. Any morphism in M can be approximated by combinations of the elements a_i, we can call these combinations words. We can group words in G according to the equivalence relation defined above, namely according to which knot they form. Thus, the set of all words written in the small set of symbols S generates an infinite subgroup which is dense in M.

Imagine that each time you open the drawer wherein lies your string, you apply a morphism which is approximated by some word in G. In fact, lets just imagine that each time we open the drawer we apply some generator $a_i \in S$. Conversely, we could say that each time we open the drawer to rummage for a glue stick, we are applying some map that takes the interval, or ring, to some knot.

One might formally look at this from a slightly higher perspective and note that we are building a model of the tangles. We will find that this particular model is very accurate and could be used to predict the most likely knot which is encountered in drawers all around the world. Nevertheless, we are still just making a model, the validity of which is entirely suspect at this point. Again, somewhat more generally, it will be useful for demonstrating that the entropy of the knot is not as solid as one might think. Also, it will provide a slightly more general way of understanding that the entropy familiar to us in thermodynamics is also related to the model which we chose.

In order to understand tangles thermodynamically, we need to assign a probability distribution to the set of possible morphisms of the string. We can do this by looking at the generators. If we open the drawer only once, and it is equally likely that we have chosen exactly one of the generators in S. Thus, if there are N generators, there are only N possible states the string may be in, and the probability of each of the N different string states is $\frac{1}{N}$. Actually, "states" are taken as knot states, so regardless of the number of generators, given that, say, only one of the generators maps the loop to a knot, then there are only two distinct states. Statistics would be

$$p_{\text{ simple loop}} = \frac{N-2}{(N-1)}$$

$$p_{\text{knot}} = \frac{1}{(N-1)}$$

If we open the drawer j times, then we can start looking at the combinations of choosing j elements from S to form a word of length j. Furthermore, we can look at the statistics of tangles by looking at the ratio of words that map the string to one of the knot classes compared to the total set knots which may be reached after j moves. Though there are infinitely many knot classes, we are only opening the drawer finitely many times and so there are only finitely many knot classes which can be reached. The statistics are straightforward we need to understand the map between knot classes K_i and words in G. We need to find the a means of calculating the equivalence class in which a given word sits.

Once we know how to calculate the number of knot classes which can be reached by a word of length j we can talk about things like entropy, since we will have a handle on our statistics. We could take some limiting process as $j \to \infty$ and end up at some sort of probability distribution over knot classes. From there we could start talking about entropy. Note that this whole process relies on which generators one chooses, as they would define the statistics. In the infinite limit, these distinctions would be irrelevant.

In order to avoid problems of ambiguity, we need to revise what M is. Also, we need to carefully consider what we mean by identifying the endpoints.

Because we may end up mapping the string to something that intersects itself, we are not mapping to the set of knots. Instead, we are mapping to the set of possible embeddings of a graph in R_3. This is the same sort of space which is considered in loop quantum gravity.

After performing a morphism defined by the generators S we still have the problem of identifying the endpoints. This ammounts to yet another morphism which can be done in any number of ways. From our perspective, we simply need to know that all the information required to identify the knot class is contained in the word of generators. This is a serious problem, since by moving the endpoints to make a knot is completely ambiguous and doing it in different ways will result in different knots.

A letter to Dr. Lee Thank you Lee for responding to my post. I think you have my basic idea in mind. To answer your question, a string is a given curve. It is a known curve embedded in some ambient space. A tangle would be a probability distribution over curves or curve types. To model the everyday tangle, it seems intuitive that the "types" should be the countable set of knot classes, since when you give a tangle a little tug (after having pulled it out of the kitchen drawer) you end up with something that, were you to identify its ends, would be a knot. Basic idea is that, since we do not By "diffeomorphism" I meant some morphism of the string (not the ambient space), like a homotopy, but with say only one end attached to a base point. When I said "group", I meant, some unknown algebra with manifold structure. By subgroup, then, I meant some sub algebra. Perhaps instead of talking about tangles, we can imagine taking a loop of string and throwing it onto the floor. At this point it is still just a loop however there are now a bunch of crossings. There is a countable set of reidemeister move-combinations that will remove all the crossings. The point is that the observer doesn't know which set of reidemeister moves to use at first. The act of throwing the string can be seen as the application of a random set of reidmeister moves. Label the reidemeister moves A, B, C (as there are three) and then each possible state is counted by equivalence classes of words in the alphabet A,B,C.

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